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GROUP THEORY.

3. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Show that the equation $x^4-ax^3+bx^2-ax+1=0$, where a and b are rational numbers, while neither $2-b+(\frac{1}{2}a)^2$ nor $(1+\frac{1}{2}b)^2-a^2$ is the square of a rational number, is irreducible in the domain R of rational numbers; determine its group for this domain.

Solution by R. L. Börger, A. B., Graduate Student, The University of Chicago.

If reducible it could assume either of the forms:

(1) $(x^2+px+1)(x^2+qx+1)=x^4+(p+q)x^3+(2+pq)x^2+(p+q)x+1=0.$

(2) $(x^2+px+r)(x^2+qx+\frac{1}{r})=x^4+(p+q)^3+(r+\frac{1}{r}+pq)x^2+(qr+\frac{p}{r})x+1=0$

From (1), equating coefficients,

$$p+q=-a\,;\ pq=b-2.\quad \therefore \left.\begin{matrix}p\\q\end{matrix}\right\}=-\tfrac{1}{2}a \pm \sqrt{(\tfrac{1}{2}a)^2+2-b}\,.$$

From (2), $p+q=-a$; $qr+\frac{p}{r}=-a$; $r+\frac{1}{r}+pq=b$. $\therefore p=-\frac{ar}{r+1}$; $q=-\frac{a}{r+1}$.

Substituting these in last we get

$$\frac{r^2+1}{r}+a^2\frac{r}{(r+1)^2}=b,\ \text{or}\ \frac{(r+1)^2}{r}+a^2\frac{r}{(r+1)^2}=b+2.\quad \text{Put}\ \frac{(r+1)^2}{r}=x.$$

Then we have,

$$x^2-(b+2)x+a^2=0,\ \text{or}\ x=\frac{b+2}{2}\pm\sqrt{\frac{(b+2)^2}{4}-a^2}\,.$$

Now if the radicals for the above two quadratics are irrational then the original equation can not be reducible. Hence the equation is irreducible if neither $(2-b)+(\frac{1}{2}a)^2$ nor $(1+\frac{1}{2}b)^2-a^2$ is the the square of a rational number.

To determine the group of the equation. Call its roots α_0, β_0, α_1, β_1 where the β's are reciprocals of the α's. Then $\alpha_0\beta_0+\alpha_1\beta_1=2$. Its conjugates under G_4 are $\alpha_1\beta_0+\alpha_0\beta_1$ and $\alpha_0\alpha_1+\beta_0\beta_1$. No two of them can be equal for then the equation would have equal roots, which is impossible because of its irreducibility. The group is then either either G_8 consisting of

$$[1,\ (\alpha_0\beta_0);\ (\alpha_1\beta_1);\ (\alpha_0\beta_0)(\alpha_1\beta_1);\ (\alpha_0\beta_1)(\alpha_1\beta_0);\ (\alpha_0\alpha_1)(\beta_0\beta_1);$$
$$(\alpha_0\alpha_1\beta_0\beta_1);\ (\alpha_1\alpha_0\beta_1\beta_0)],\ \text{or}$$

(a) 1; $(\alpha_0\beta_0)$; $(\alpha_1\beta_1)$; $(\alpha_0\beta_0)(\alpha_1\beta_1)$,

(b) 1; $(\alpha_0\beta_0)(\alpha_1\beta_1)$; $(\alpha_0\alpha_1\beta_0\beta_1)$; $(\alpha_1\alpha_0\beta_1\beta_0)$,

(c) 1; $(\alpha_0\beta_0)(\alpha_1\beta_1)$; $(\alpha_0\alpha_1)(\beta_0\beta_1)$; $(\alpha_0\beta_1)(\alpha_1\beta_0)$.

The group must be transitive since the equation is irreducible. Hence we may exclude (a).

It cannot be (b) since (b) is the regular cyclic group, and the equation, if this were its group, would be Abelian.

We compute $(\alpha_0-\beta_0)(\alpha_1-\beta_1)$ which belongs to the group (c):

$$\alpha_0\beta_0+\alpha_1\beta_1+\alpha_0\beta_1+\alpha_1\beta_0+\alpha_0\alpha_1=b.$$

$\therefore \alpha_0\beta_1+\alpha_1\beta_0+\alpha_0\alpha_1+\beta_0\beta_1=b-2.$

Factoring this, $(\alpha_0+\beta_0)(\beta_1+\beta_1)=b-2$. Squaring, and remembering that $\alpha_0\beta_0=\alpha_1\beta_1=1$,

$$(\alpha_0-\beta_0)^2(\alpha_1-\beta_1)^2=[(\alpha_0+\beta_0)^2-4][(\alpha_1+\beta_1)^2-4].$$

$\therefore (\alpha_0-\beta_0)(\alpha_1-\beta_1)=2\sqrt{[(1+\frac{1}{2}b)^2-a^2]}$, and as the irrationality of the radical is one of the conditions for irreducibility this group cannot be the group of the equation. The group of the equation is therefore G_8.

PROBLEMS FOR SOLUTION.

ALGEBRA.

208. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College Defiance, O.

Solve $x^4+y^4=14x^2y^2$; $x+y=m$.

209. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Prove that $(a^4+b^4+c^4+d^4)>4abcd$.

210. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

The sum of five quantities and the sum of their cubes are both zero. Show that the sum of their fifth powers is a factor of the sum of any odd powers of the quantities.

GEOMETRY.

236. Proposed by J. R. HITT.

If two sides of a triangle pass through a fixed point, the third side touches a fixed circle.